Integration of the Lorentz-Dirac equation: Interaction of an intense laser pulse with high-energy electrons

James Koga*
Advanced Photon Research Center, Kansai Research Establishment, Japan Atomic Energy Research Institute, 8-1 Umemidai, Kizu, Souraku, Kyoto 619-0215, Japan
(Received 22 October 2003; revised manuscript received 4 June 2004; published 29 October 2004)

Usually the motion of an electron under the influence of electromagnetic fields is influenced to a small extent by radiation damping. With the advent of high power high irradiance lasers it has become possible to generate focused laser irradiances where electrons interacting with the laser become highly relativistic over very short time and spatial scales. By focusing petawatt class lasers to very small spot sizes the amount of radiation emitted by electrons can become very large. Resultingly, the damping of the electron motion by the emission of this radiation can become large. In order to study this problem a code is written to solve a set of equations describing the evolution of a strong electromagnetic wave interacting with a single electron. Usually the equation of motion of an electron including radiation damping under the influence of electromagnetic fields is derived from the Lorentz-Dirac equation treating the damping as a perturbation. We use this equation to integrate forward in time and use the Lorentz-Dirac equation to integrate backward in time. We show that for very short wavelength electromagnetic radiation deep in the quantum regime at high irradiances differences between the perturbation equation and Lorentz-Dirac can be seen. However, for electron motion in the classical regime the differences are negligible. For electron motion in the classical regime the first order damping equation is found to be very adequate.

DOI: 10.1103/PhysRevE.70.046502 PACS number(s): 41.60.−m, 41.75.Ht, 14.60.Cd

I. INTRODUCTION

With the advent of petawatt level high power short pulse lasers it may become possible via strong focusing to extend the irradiance to levels close to 10^{22} W/cm^2 [1]. At such an irradiance electrons can theoretically reach high energies. Under such extreme conditions the effect of radiation damping on the electron motion in the intense wave can become large [2]. In addition, by using counter-propagating laser pulses irradiances reaching the Schwinger limit, 10^{29} W/cm^2 could be achieved with current lasers [3]. Taking into account coherent radiation effects the damping could be strong even for relatively low irradiance laser pulses interacting with clusters [4]. Under most conditions here on the Earth radiation damping is usually a small perturbation. Even with current high energy storage rings the strength of the damping is small [5]. So radiation damping has been verified only under conditions where it is small. By using high irradiance lasers we can examine the dynamics of electrons under strong damping conditions. In this paper we examine the effects of radiation damping on a single electron under the influence of a strong electromagnetic wave. The equations of motion describing the damping are solved numerically. With the advent of ultra-high irradiance lasers it has become possible to probe the boundaries of classical electrodynamics.

II. RADIATION DAMPING

Radiation damping or radiation reaction occurs when an electron is accelerated. When the electron is accelerated, it emits radiation. This radiation causes the electron to lose its energy. Under most circumstances the amount of radiation emitted is very small and represents a small perturbation to the electron motion. However, this is an important problem. Radiation damping limits the maximum energy which electrons can be economically stored in a storage ring due to energy losses from the radiation emitted [5]. The problem of radiation damping was one of the causes for the development of quantum theory. In the classical model of an atom an electron circulating around the nucleus would lose energy so that it would eventually spiral in to the center. Nonrelativistic equations describing the effects of radiation damping have been around for nearly a hundred years [6,7]. The relativistically covariant form of the equation of motion of a radiating electron was first derived by Dirac [8]:

$$\frac{du^i}{ds} = \frac{e}{mc^2}F^{ik}u_k + \frac{2e^2}{3mc^2}g^i, \tag{1}$$

$$g^i = \left[\frac{d^2u^i}{ds^2} - u^i u^k \frac{d^2u_k}{ds^2}\right], \tag{2}$$

where u^i is the four velocity, g^i is the damping term, and F^{ik} is the electromagnetic field tensor.

This equation is sufficient to describe the damping of a relativistic electron interacting with an electromagnetic field. However, from a mathematical as well as numerical point of view there are difficulties. Since there are second order derivatives in the velocity, three initial conditions are necessary to solve the equations of motion. If the equations are integrated forward in time, there is an exponential blowup in the energy of the electron even if there is no electromagnetic field present. This problem deals with some of the underlying

*Electronic address: koga@apr.jaeri.go.jp; URL: http://wwwapr.apr.jaeri.go.jp/aprc/e/index_e.html

inconsistencies of electromagnetic theory [9]. In order to avoid this problem the equations are integrated backwards in time [10] and the acceleration is required to vanish at infinity [11]. However, for numerical simulation purposes this is difficult as we need to know the final conditions in order to do this. Another option is to re-express the damping term in terms of the fields [9]. This can be done through a perturbation expansion of the equation of motion. If we assume that the damping term g^i in Eq. (1) is a small perturbation, then to zeroth order the acceleration can be described by [9]

$$\left(\frac{du^i}{ds}\right)_0 = \frac{e}{mc^2}F^{ik}u_k, \tag{3}$$

where we have used the subscript 0 to specify the zeroth order term. The equation with damping is then included by using this equation in the damping term g^i in Eq. (2):

$$\left(\frac{du^i}{ds}\right)_1 = \left(\frac{du^i}{ds}\right)_0 + \frac{2e^2}{3mc^2}g_0^i, \tag{4}$$

$$g_0^i = \left[\left(\frac{d^2u^i}{ds^2}\right)_0 - u^iu^k\left(\frac{d^2u_k}{ds^2}\right)_0\right], \tag{5}$$

where the subscript 1 refers to the first order term. The damping force is now expressed in terms of electromagnetic fields only. The resulting equation can now be integrated forward in time. Writing everything in terms of fields for the damping force we get [9]

$$g_0^i = \left[\frac{e}{mc^2}\frac{\partial F^{ik}}{\partial x^l}u^lu_k + \left(\frac{e}{mc^2}\right)^2 F^{ik}F_{kl}u^l - \left(\frac{e}{mc^2}\right)^2 (F_{km}u^k)(F^{ml}u_l)u^i\right]. \tag{6}$$

Explicitly the equation of motion becomes

$$\frac{d\gamma\boldsymbol{\beta}}{dt} = \frac{e}{mc}(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) + \frac{2e^2}{3mc^2}\mathbf{g}_0, \tag{7}$$

and the explicit expression for the spatial part of the damping force is [9]

$$\mathbf{g}_0 = \frac{e}{mc^2}\gamma\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\boldsymbol{\nabla}\right)(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B}) + \left(\frac{e}{mc^2}\right)^2 c[(\boldsymbol{\beta}\cdot\mathbf{E})\mathbf{E} + (\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B})\times\mathbf{B}] - \left(\frac{e}{mc^2}\right)^2\gamma^2 c\boldsymbol{\beta}[(\mathbf{E} + \boldsymbol{\beta}\times\mathbf{B})^2 - (\boldsymbol{\beta}\cdot\mathbf{E})^2]. \tag{8}$$

The Lorentz-Dirac equation is the equation of motion for a point particle. This first order equation, the so called Landau-Lifshitz equation, has been both proposed as the exact equation of motion for a point particle [12] and the equation of motion for an electron with structure [13]. This equation avoids the preacceleration problem of the Lorentz-Dirac equation, however, suffers from a very small departure from the correct rate of radiation [14].

Higher order terms can be obtained by continuing the previous procedure. We can go to second order to obtain

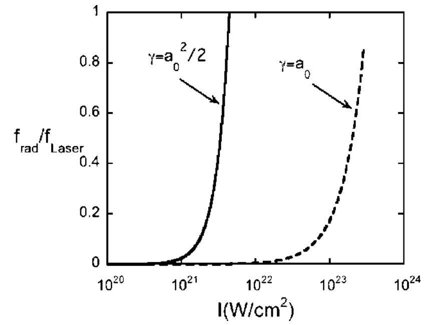

FIG. 1. The ratio between the damping force f_{RD} and Lorentz force f_L versus irradiance for different scalings of the electron energy.

$$\left(\frac{du^i}{ds}\right)_2 = \left(\frac{du^i}{ds}\right)_1 + \frac{2e^2}{3mc^2}g_1^i, \tag{9}$$

$$g_1^i = \left[\left(\frac{d^2u^i}{ds^2}\right)_1 - u^iu^k\left(\frac{d^2u_k}{ds^2}\right)_1\right]. \tag{10}$$

Re-expressing this second order equation in terms of just the fields we get

$$\left(\frac{du^i}{ds}\right)_2 = \frac{e}{mc^2}F^{ik}u_k + \frac{2e^2}{3mc^2}g_0^i + \left(\frac{2e^2}{3mc^2}\right)^2 g_1^i, \tag{11}$$

where g_0^i is from Eq. (6) and g_1^i is a large expression containing second order corrections. From Eq. (11) we can see that the second order terms are smaller than the first order terms by a factor of r_e where r_e is the classical electron radius e^2/mc^2.

We can estimate the regime where radiation damping becomes large compared to the Lorentz force f_L by estimating the strength of the largest damping term, the third term, of $\mathbf{f}_{RD}$ in Eq. (8) [2,9]:

$$f_{RD} \sim \frac{2e^2\omega_0^2\gamma^2a_0^2}{3c^2}, \tag{12}$$

$$f_L = mc\omega_0 a_0, \tag{13}$$

where ω_0 is the laser frequency, γ is the relativistic factor of the electron, and a_0 is the normalized laser amplitude $eE_0/mc\omega_0$ where E_0 is the peak laser amplitude. The ratio is

$$\frac{f_{RD}}{f_L} \sim \frac{2}{3}\frac{r_e\omega_0}{c}\gamma^2 a_0. \tag{14}$$

Assuming an initially stationary electron's energy increases as $a_0^2/2$ and counter-propagates with respect to the laser pulse, the radiation damping force can be significant at laser irradiances above 5×10^{21} W/cm^2 for a laser of wavelength 0.88 μm. Figure 1 shows a plot of the ratio f_{RD}/f_L verses irradiance assuming the initially stationary electron energy scales as $a_0^2/2$ and a_0. The different scalings in the electron energy is due to the fact that in laser-plasma interactions the energy can scale differently from the ideal plane wave result of $a_0^2/2$. We can see from the figure that at irradiances above roughly 10^{22} W/cm^2 the damping can become significant even for the a_0 scaling. This has been shown by particle-in-cell simulations incorporating the largest term in the radiation reaction force in Eq. (6) where damping was found to become significant above $5\times10^{22}\ \mathrm{W/cm^2}$ [2]. Different scalings can occur in f_{RD}/f_L due to mostly co-propagation of the electrons, collective plasma effects, radiation effects, and quantum effects [4].

It can be seen in Eq. (14) that the damping can become large when $\gamma\gg1$, $a_0\gg1$, and the laser wavelength is $\lambda_0 \sim O(r_e)$. In these regimes the assumption that the damping is a perturbation may be limited. Since the damping can become significant, we examine whether higher order terms in the expansion are necessary in the description of the damping.

In order to investigate the regimes where higher order effects might play a role, we additionally use the Lorentz-Dirac equations of motion, Eqs. (1) and (2). We first integrate forward in time using the first order damping equation, Eqs. (7) and (8). After the electron has passed through the laser pulse we use the final position, velocity, and acceleration as the initial conditions for the Lorentz-Dirac equation and integrate backwards in time. Rewriting Eqs. (1) and (2) we get

$$\frac{d^2u^i}{ds^2} = \left(\frac{2e^2}{3mc^2}\right)^{-1}\left(\frac{du^i}{ds} - \frac{e}{mc^2}F^{ik}u_k\right) + u^i\frac{u^k}{ds}\frac{du_k}{ds}. \tag{15}$$

Large differences between the initial conditions of the electron and the final condition of the backwards integrated electron will indicate that higher order effects are needed.

Since higher order terms are smaller than the first order terms by a factor of r_e, significant effects may occur when the wavelength of the radiation is short $\lambda_0 \sim O(r_e)$.

However, when the wavelength of the radiation becomes short we must take care about quantum kinematic effects. These effects become important when the wavelength of the radiation becomes comparable to the Compton wavelength of the electron in it's rest frame $\lambda_0 \sim \lambda_C$ where $\lambda_C = h/m_e c$. Using Doppler shift formulas this condition can be rewritten as

$$\lambda_0 \geqslant \lambda_C \gamma(1-\beta\cos\theta), \tag{16}$$

where θ is the angle between the radiation propagation direction and the direction of the electron. For counter-propagating lasers and electrons this becomes

$$\lambda_0 \geqslant \lambda_C\gamma(1+\beta) \approx 2\gamma\lambda_C, \tag{17}$$

and for co-propagating lasers and electrons this becomes

$$\lambda_0 \geqslant \lambda_C\gamma(1-\beta) \approx \frac{\lambda_C}{2\gamma}. \tag{18}$$

For high energy electrons $(\gamma\gg1)$ only long wavelengths are suitable in the classical description for counter-propagation. In contrast, for co-propagation the restriction is eased and we can go to short wavelengths.

III. RESULTS

We calculated the interaction of a single electron with a Gaussian laser pulse of the form

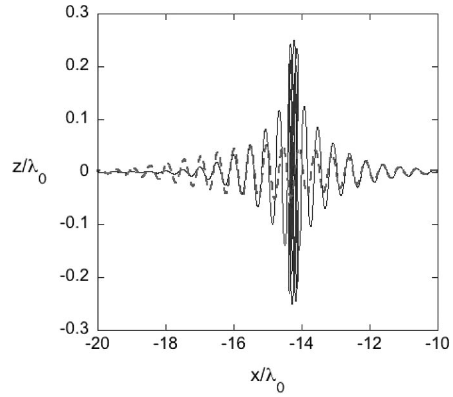

FIG. 2. Trace of the electron motion in the *x*-*z* plane where the electron is propagating to the left and the laser is propagating to the right. The solid line is for damping and the dotted line is no damping.

$$\mathbf{E}(\mathbf{x},t) = \hat{z}E_0 h(\phi)\sin(\phi), \tag{19}$$

$$\mathbf{B}(\mathbf{x},t) = -\hat{y}E_0 h(\phi)\sin(\phi), \tag{20}$$

where $\phi=\omega_0(t-x/c)$, ω_0 is the laser frequency,

$$h(\phi) = \exp\left[-\left(\frac{\phi}{\omega_0\Delta\tau}\right)^2\right], \tag{21}$$

and $\Delta\tau$ is the pulse width. A Gaussian pulse was chosen to assure that the acceleration far enough away from the laser pulse approaches zero. This is the asymptotic condition where the acceleration is required to vanish after a long time which ensures a unique solution to the Lorentz-Dirac equation [11].

Equation (7) with Eqs. (8) and (15) are integrated in time using an adaptive Runge-Kutta integration scheme [15].

Figure 2 shows the interaction of a laser pulse of irradiance $5\times10^{22}\ \mathrm{W/cm^2}$ with a wavelength of $\lambda_0=1\ \mu$m and pulse width of $\Delta\tau=20$ fs counter propagating with an electron of energy 150 MeV. These are parameters similar to those in previous one dimensional simulations [2]. The laser pulse is propagating from the left to the right and the electron, which starts at x/λ_0 and z/λ_0 equals zero, is propagating from the right to the left. The figure shows the trace of the electron's trajectory with damping (solid line) and without damping (dotted line). It can be seen that the oscillations of the electron in the direction of the laser electric field are larger than that without damping. This is due to the fact that the electron is losing energy as it propagates due to radiation damping. The amount of energy lost can be seen in Fig. 3. In the figure γ is plotted versus x/λ_0. We can see that in the case of damping more than 80% of the electron's energy is lost in the form of radiation. However, when we compare the difference between the first order damping and Lorentz-Dirac backward integrations the differences between the particle motion are insignificant. This can be attributed to the fact that the Doppler shifted wavelength of the laser is large compared to the classical electron radius. The higher order

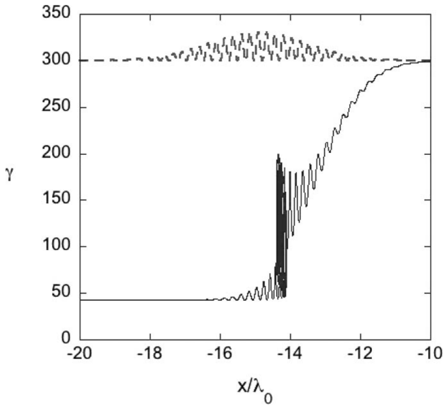

FIG. 3. The trace of the electron motion in the x-γ plane where the electron is propagating to the left and the laser is propagating to the right. The solid line is for damping and the dotted line is no damping.

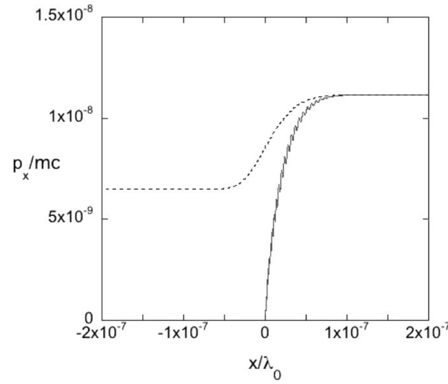

FIG. 5. The trace of the electron motion in the x-p_x plane where the electron is initially stationary and the laser is propagating in the x direction before they interact. The solid line is for first order damping and the dotted line is for the Lorentz-Dirac equation. The laser irradiance is $5.5\times 10^{22}\ \mathrm{W/cm^2}$ with $\lambda_0 = 10^{-12}$ cm and laser pulse duration of $\Delta\tau = 2\times 10^{-22}$ s.

corrections to the radiation reaction force are smaller than the first order terms by a factor of the classical electron radius.

At very short wavelengths differences between the first order and Lorentz-Dirac equations of motion were found, but only for $\lambda_0 \ll \lambda_C$ which is far into the quantum regime where these classical equations are of questionable validity. For $\lambda_0 > \lambda_C$ no significant differences between the first order and Lorentz-Dirac equations were found.

Figure 4 shows the x-z motion of an initially stationary electron interacting with a laser of irradiance $5.5\times 10^{22}\ \mathrm{W/cm^2}$, pulse duration $\Delta\tau = 2\times 10^{-22}$ seconds, and wavelength $\lambda_0 = 10^{-12}$ cm where $a_0 = 2\times 10^{-5}$. Integrated particle motions for the first order damping and Lorentz-Dirac equations of motion are indicated by the solid and dotted lines, respectively. The laser pulse is propagating from the left to the right and the electron starts at x/λ_0 and z/λ_0 equals zero. Figure 5 shows the x-p_x motion of the electron where the solid and dotted lines refer to the first order damping and Lorentz-Dirac equation solutions, respectively. It can be seen that the electron motion spans only a very small fraction of the laser wavelength.

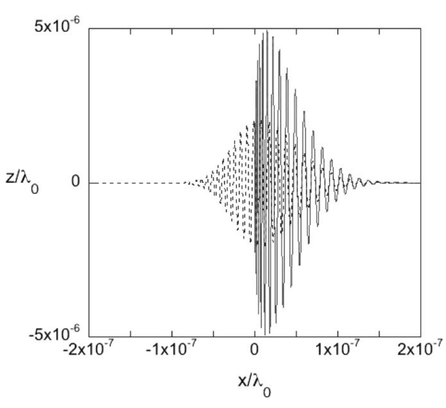

FIG. 4. Trace of the electron motion in the x-z plane where the electron is initially stationary and the laser is propagating in the x direction before they interact. The solid line is for first order damping and the dotted line is for the Lorentz-Dirac equation. The laser irradiance is $5.5\times 10^{22}\ \mathrm{W/cm^2}$ with $\lambda_0 = 10^{-12}$ cm and laser pulse duration of $\Delta\tau = 2\times 10^{-22}$ s.

It can be seen that in the case of the Lorentz-Dirac equation that the electron which was initially stationary in the case of the first order damping equation has an initial momentum in the $+x$ direction. When we examine the simulation results, we see that the electron which is pushed using the first order damping equation of motion is accelerated more strongly by the laser pulse. From a physical point of view this seems to indicate that the cross section in this case is larger than in the Lorentz-Dirac equation. This would support the view that the first order damping equation of motion gives the electron structure beyond that of a point particle. Since these parameters are deep within the quantum regime, both equations of motion are of questionable applicability. However, from a theoretical point of view it does shed light on the different characteristics of the equations of motion.

IV. CONCLUSIONS

We have performed numerical calculations in the case of large damping in the motion of an electron in a very strong laser pulse. By using both the first order damping equation and the Lorentz-Dirac equation of motion we have shown that higher order damping effects do not play a role in the classical regime ($\lambda_0 > \lambda_C$). However, deep into the quantum regime ($\lambda_0 \approx r_e \ll \lambda_C$) higher order damping effects become apparent. Since this occurs in the quantum regime, the validity of the damping equations are questionable. For electron motion in the classical regime the first order damping equation is found to be very adequate.

Although the electron has been measured in accelerator experiments to have a size r much smaller than the classical electron radius ($r \leqslant 2.8 \times 10^{-17}\ \mathrm{cm} \ll r_e$) [16], we may be able to study the electron field structure by performing such an experiment with counter-propagating laser pulses and high energy electrons. High irradiance lasers present a way to study the radiation damping of electrons under strong damping conditions. By going to very short wavelengths such lasers could aid in the study of the electron field structure near the classical electron radius.

ACKNOWLEDGMENTS

We wish to acknowledge fruitful comments and suggestions from A. Zhidkov, S. V. Bulanov, T. Esirkepov, M. Yamagiwa, K. Mima, and T. Tajima. This work was partially performed under the Grant-in-Aid for Specially Promoted Research (Contract No. 15002013) and the Grant-in-Aid for Creative Scientific Research (Contract No. 15GS0214) of the Ministry of Education, Culture, Sports, Science and Technology of Japan (MEXT).

[1] M. Aoyama, K. Yamakawa, Y. Akahane, J. Ma, N. Inoue, H. Ueda, and H. Kiriyama, Opt. Lett. **28**, 1594 (2003).

[2] A. Zhidkov, J. Koga, A. Sasaki, and M. Uesaka, Phys. Rev. Lett. **88**, 185002 (2002).

[3] S. V. Bulanov, T. Esirkepov, and T. Tajima, Phys. Rev. Lett. **91**, 085001 (2003).

[4] S. V. Bulanov, T. Esirkepov, J. Koga, and T. Tajima, Plasma Phys. Rep. **30**, 196 (2004).

[5] H. Wiedemann, *Particle Accelerator Physics I* (Springer-Verlag, Berlin, 1999).

[6] M. Abraham, *Theorie der Elektrizitat, Vol II: Elektromagnetische Theorie der Strahlung* (Teubner, Leipzig, 1905).

[7] H. A. Lorentz, *The Theory of Electrons* (Teubner, Leipzig, 1909).

[8] P. A. M. Dirac, Proc. R. Soc. London, Ser. A **167**, 148 (1938).

[9] L. D. Landau and E. M. Lifshits, *The Classical Theory of Fields* (Pergamon, New York, 1994).

[10] F. V. Hartemann and A. K. Kerman, Phys. Rev. Lett. **76**, 624 (1996).

[11] F. Rohrlich, *Classical Charged Particles* (Perseus Books Publishing, Reading, MA, 1965).

[12] F. Rohrlich, Phys. Lett. A **283**, 276 (2001).

[13] R. F. O'Connell, Phys. Lett. A **313**, 491 (2003).

[14] R. Rivera and D. Villarroel, Phys. Rev. E **66**, 046618 (2002).

[15] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes in FORTRAN* (Cambridge University Press, Cambridge, 1992).

[16] D. Bourilkov, Phys. Rev. D **62**, 076005 (2000).